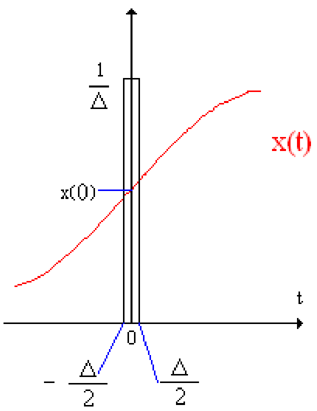
Esantionarea semnalelor

Discretizarea variatiei in timp a semnalului.

Toerema esantionarii

Esantionarea ideala



$$u_\Delta(t) = \frac{1}{\Delta}\left[\sigma\left(t+\frac{\Delta}{2}\right) - \sigma\left(t-\frac{\Delta}{2}\right)\right]$$

$$x(t)u_\Delta(t) \cong x(0)u_\Delta(t)$$

$$x(t)u_\Delta(t-kT_e) \cong x(kT_e)u_\Delta(t-kT_e)$$

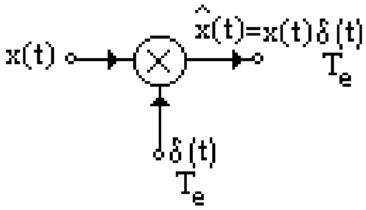
$$x(t)\sum_{k=-\infty}^{\infty} u_\Delta(t-kT_e) \cong \sum_{k=-\infty}^{\infty} x(kT_e)u_\Delta(t-kT_e)$$

$$\lim_{\Delta\to 0} u_\Delta(t) = \delta(t)\ ;$$

$$\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} u_\Delta(t-kT_e) = \sum_{k=-\infty}^{\infty} \delta(t-kT_e) = \delta_{T_e}(t)$$

$$\hat{x}(t) = x(t)\delta_{T_e}(t) = \sum_{k=-\infty}^{\infty} x(kT_e)\delta(t-kT_e)$$

$$\hat{x}(t) = x(t)\delta_{T_e}(t) = \sum_{k=-\infty}^{\infty} x(kT_e)\delta(t - kT_e)$$



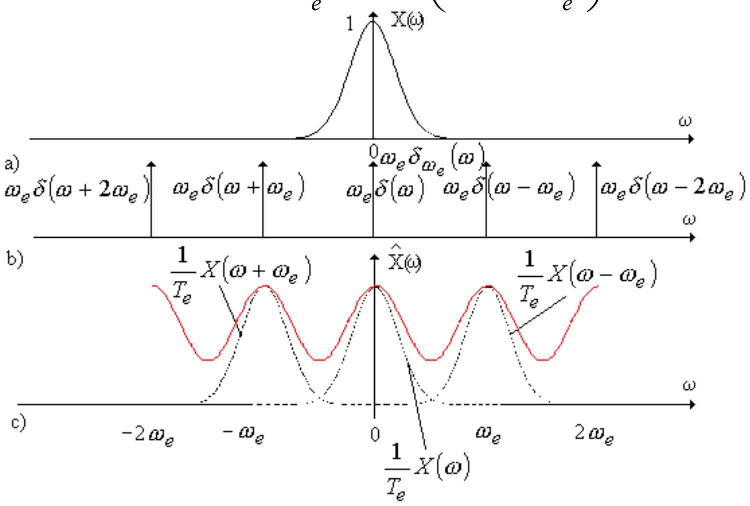
Spectrul semnalului esantionat ideal

$$\delta_{T_e}(t) \leftrightarrow \frac{2\pi}{T_e}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_e}\right); \frac{2\pi}{T_e} = \omega_e$$

$$\hat{X}(\omega) = \mathcal{F}\{x(t)\delta_{T_e}(t)\} = \frac{1}{2\pi}X(\omega) * \frac{2\pi}{T_e}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T_e}\right) =$$

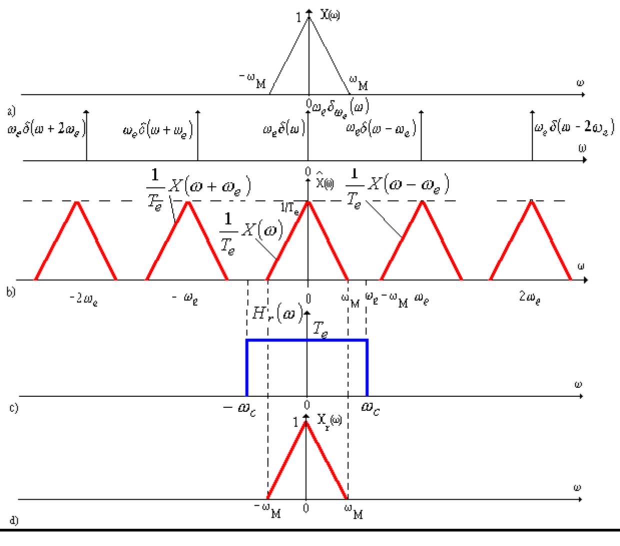
$$= \frac{1}{T_e}\sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\omega - k\frac{2\pi}{T_e}\right) = \frac{1}{T_e}\sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T_e}\right)$$

$$\hat{X}(\omega)=\frac{1}{T_e}\sum_{k=-\infty}^{\infty}X\left(\omega-k\frac{2\pi}{T_e}\right)$$



Eroarea de aliere.

Teorema esantionarii semnalelor de banda limitata



$\omega_e > 2\omega_M$

$\omega_M \le \omega_c \le \omega_e - \omega_M$

Nu apare aliere.

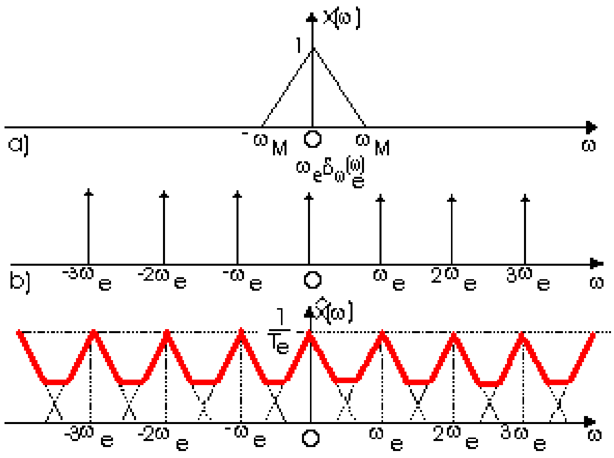
$|H_r(0)| = T_e$

$$H_r(\omega) = T_e p_{\omega_c}(\omega) = \begin{cases} T_e, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \quad \omega_M \le \omega_c \le \omega_e - \omega_M$$

$$x_r(t) = \hat{x}(t) * h_r(t) \leftrightarrow X_r(\omega) = \hat{X}(\omega) \cdot H_r(\omega) =$$

$$= \frac{1}{T_e} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_e) T_e p_{\omega_c}(\omega) = X(\omega),$$

$x_r(t) = x(t)$, a.p.t



$$\omega_e - \omega_M < \omega_M$$

Apare alierea.

$$H_r(\omega) = T_e p_{\omega_c}(\omega) \leftrightarrow h_r(t) = T_e \frac{sin\,\omega_c t}{\pi t}$$

$$x_r(t) = h_r(t) * \hat{x}(t) = T_e \frac{sin\,\omega_c t}{\pi t} * \sum_{k=-\infty}^{\infty} x(kT_e)\delta(t - kT_e) =$$

$$= \sum_{k=-\infty}^{\infty} x(kT_e)T_e \frac{sin\,\omega_c t}{\pi t} * \delta(t - kT_e) = \sum_{k=-\infty}^{\infty} x(kT_e)T_e \frac{sin\,\omega_c (t - kT_e)}{\pi(t - kT_e)} =$$

$$= \sum_{k=-\infty}^{\infty} \frac{2\omega_c}{\omega_e} x(kT_e) \frac{sin\,\omega_c (t - kT_e)}{\omega_c (t - kT_e)}$$

Frecventa de esantionare minima este $\omega_e = 2\omega_M$ si poarta denumirea de frecventa de esantionare Nyquist. In cazul esantionarii la frecventa Nyquist formula de reconstructie devine: $x_r(t) = \sum_{k=-\infty}^{\infty} x(kT_e) \frac{sin\,\omega_M (t - kT_e)}{\omega_M (t - kT_e)}$

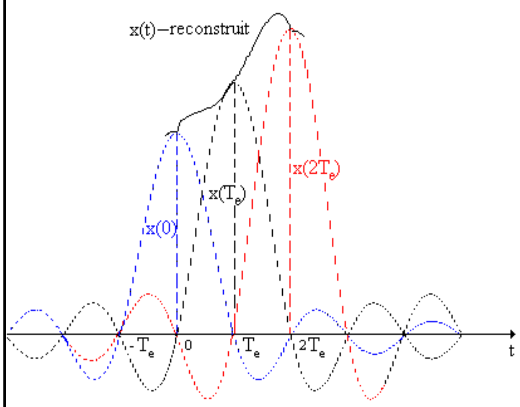
Teorema WKS (Whittaker, Kotelnicov, Shannon)

Daca semnalul $x(t)$ este de banda limitata la ω_M, in sensul ca $X(\omega) \equiv 0$ pentru $|\omega| > \omega_M$, atunci $x(t)$ este unic determinat de multimea esantioanelor sale $\{x(nT_e) | n \in Z\}$, daca $\omega_e \geq 2\omega_M$, adica frecventa de esantionare este cel putin dublul frecventei maxime. In conditiile de mai sus semnalul initial $x(t)$ se poate reconstitui din esantioanele sale, a.p.t prin relatia:

$$x(t) = \sum_{k=-\infty}^{\infty} x(kT_e) \frac{2\omega_c}{\omega_e} \frac{sin\,\omega_c (t - kT_e)}{\omega_c (t - kT_e)}$$

cu conditia ca ω_c sa fie astfel ales incat sa satisfaca relatia: $\omega_M \leq \omega_c \leq \omega_e - \omega_M$.

Reconstructia prin filtrare trece-jos ideala



$$x(t)=\sum_{k=-\infty}^{\infty} x(kT_e)\frac{2\omega_c}{\omega_e}\frac{\sin\omega_c(t-kT_e)}{\omega_c(t-kT_e)}$$

$$x(nT_e)=\sum_{k=-\infty}^{\infty} x(kT_e)\frac{2\omega_c}{\omega_e}\frac{\sin\omega_c(nT_e-kT_e)}{\omega_c(nT_e-kT_e)}$$

$$\omega_M=\frac{\omega_e}{2}\Rightarrow \omega_M T_e=\pi$$

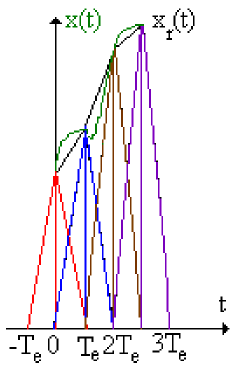
$$x(nT_e)=\sum_{k=-\infty}^{\infty} x(kT_e)\frac{\sin\pi(n-k)}{\pi(n-k)}=$$

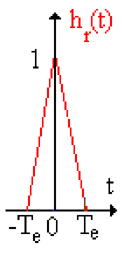
$$=\sum_{k=-\infty}^{\infty} x(kT_e)\delta_{n,k}=x(nT_e)$$

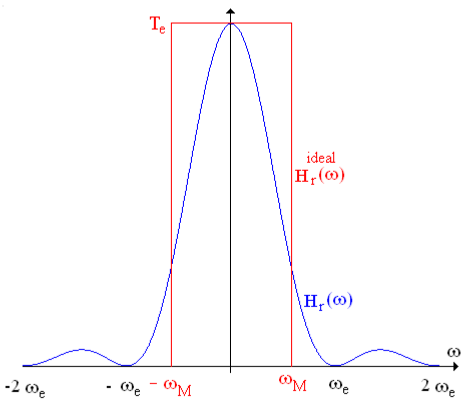
$$\delta_{n,k}=\begin{cases}1, & n=k\\ 0, & n\neq k\end{cases}$$

Reconstructia prin interpolare

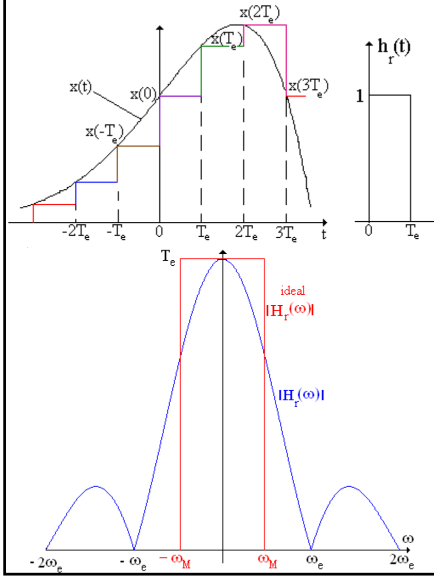
$$H_r(\omega)=T_e\left(\frac{\sin\frac{\omega T_e}{2}}{\frac{\omega T_e}{2}}\right)^2$$







Reconstructia prin extrapolare de ordinul zero



$$h_r(t) = p_{\frac{T_e}{2}}\left(t - \frac{T_e}{2}\right)$$

$$\leftrightarrow e^{-j\frac{\omega T_e}{2}} \frac{2\sin\frac{\omega T_e}{2}}{\omega} =$$

$$= e^{-j\frac{\omega T_e}{2}} T_e \frac{\sin\frac{\omega T_e}{2}}{\frac{\omega T_e}{2}}$$

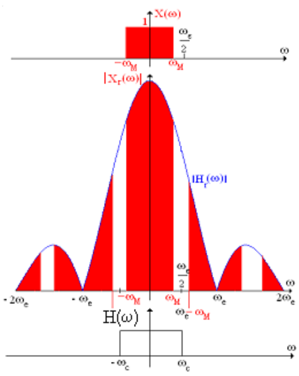
$$H_r(\omega) = e^{-j\frac{\omega T_e}{2}} T_e \frac{\sin\frac{\omega T_e}{2}}{\frac{\omega T_e}{2}} =$$

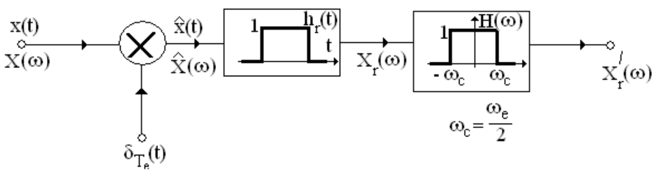
$$= e^{-j\pi\frac{\omega}{\omega_e}} \frac{\sin\pi\frac{\omega}{\omega_e}}{\pi\frac{\omega}{\omega_e}}$$

Spetrul semnalului reconstruit:

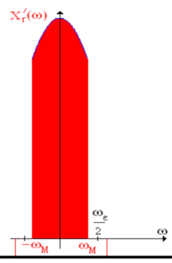
$$X_r(\omega) = \widehat{X}(\omega) H_r(\omega) = \left(\frac{1}{T_e}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_e)\right) T_e e^{-j\pi\frac{\omega}{\omega_e}} \frac{\sin\pi\frac{\omega}{\omega_e}}{\pi\frac{\omega}{\omega_e}} =$$

$$= \sum_{k=-\infty}^{\infty} e^{-j\pi\frac{\omega}{\omega_e}} \frac{\sin\pi\frac{\omega}{\omega_e}}{\pi\frac{\omega}{\omega_e}} X(\omega - k\omega_e)$$



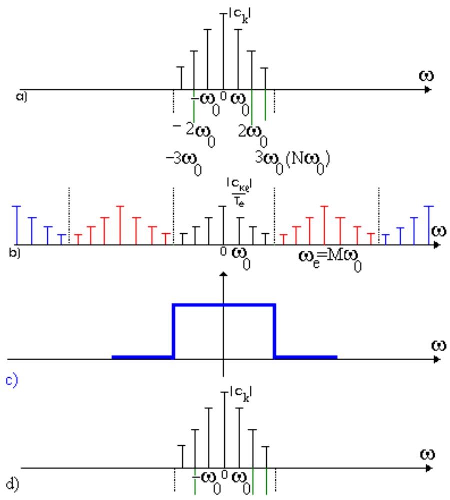


$$\omega_c = \frac{\omega_e}{2}$$



Eroarea scade daca $\omega_e \gg \omega_M$.

Esantionarea ideala a semnalelor periodice



$\omega_M = N\omega_0$; $\omega_0 = \frac{2\pi}{T_0}$; $\omega_e = M\omega_0$

Pentru ca sa nu apara suprapunerea lobilor centrali este necesar ca:

$N\omega_0 < \omega_e - N\omega_0 = \omega_0(M-N)$

Diferenta dintre $\omega_0(M-N)$ si $N\omega_0$ trebuie sa fie de forma:

$\omega_0(M-N) - N\omega_0 = R\omega_0$, R=1,2,...

sau

$\omega_e = M\omega_0 = (2N+R)\omega_0$

adica

$\omega_e = (2N+R)\omega_0 = 2\omega_M + R\omega_0$; R=1,2,...

$$H_r(\omega) = T_e p_{\omega_c}(\omega) = \begin{cases} T_e, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \; ; \; N\omega_0 < \omega_c < \omega_e - N\omega_0$$

Pentru a evita aparitia erorilor de aliere este necesar ca:

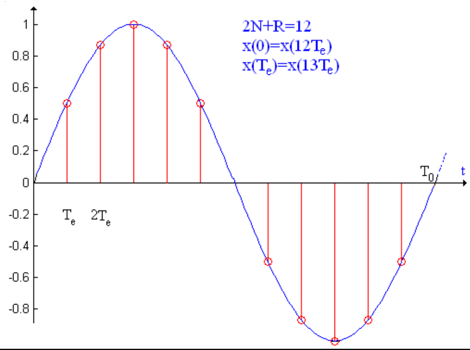
$\omega_e - N\omega_0 > N\omega_0$ sau $\omega_e > 2N\omega_0 = 2\omega_M$

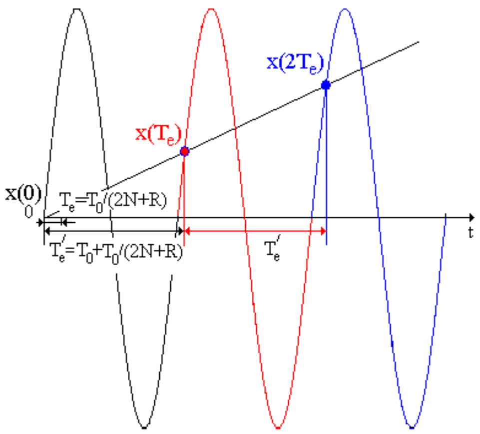
Spre deosebire de semnalele aperiodice unde $\omega_e \ge 2\omega_M$, pentru semnalele periodice trebuie sa esantionam astfel incat $\omega_e > 2\omega_M$. Pe perioada celei mai rapide componente spectrale trebuie sa prelevam mai mult de doua esantioane (adica cel putin 3).

Daca T_0 este perioada fundamentalei si daca esantionarea se face conform relatiei $\omega_e = (2N+R)\omega_0$ atunci $\frac{2\pi}{T_e} = (2N+R)\frac{2\pi}{T_0}$;

R=1,2,...sau $T_e = \frac{T_0}{2N+R}$

Doar 2N+R esantioane pot fi distincte ca urmare a periodicitatii semnalului supus esantionarii. Toate pot fi prelevate intr-o singura perioada a fundamentalei T_0.





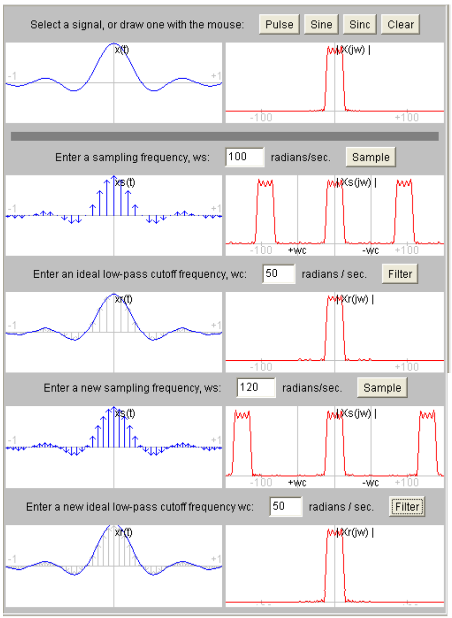
Acelasi rezultat se poate obtine si preluand esantioane succesive din perioade succesive.

$$x(kT_e) = x(T_0 + kT_e) = x(kT_0 + kT_e)$$

$$T_e' = kT_0 + T_e = kT_0 + \frac{T_0}{2N+R}$$

Aceasta posibilitate este valorificata in constructia osciloscoapelor cu esantionare.

http://www.jhu.edu/~signals/sampling/index.html



Relatii energetice

Pentru semnale aperiodice esantionate este adevarata relatia de tip Rayleigh:

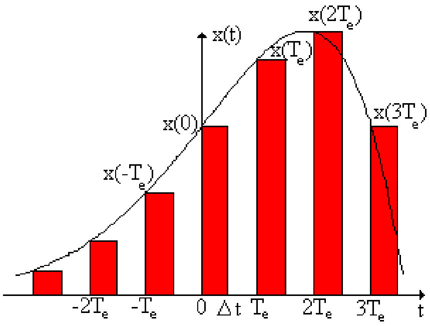
$$W = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = T_e \sum_{k=-\infty}^{\infty} \left|x(kT_e)\right|^2$$

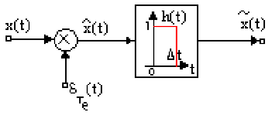
Pentru semnale periodice esantionate este valabila relatia de tip Parseval:

$$P = \frac{1}{T_0} \int_{T_0} \left|x(t)\right|^2 dt = \frac{1}{M} \sum_{k=0}^{M-1} \left|x(kT_e)\right|^2 \quad ;\ M=2N+R,\ R=1,2,...$$

Energia sau puterea pot fi calculate fie din forma de variatie in timp fie in domeniul frecventa.

Esantionarea cu memorare





$$\tilde{x}(t)=\left[x(t)\delta_{T_e}(t)\right]*h(t)=\hat{x}(t)*h(t)$$

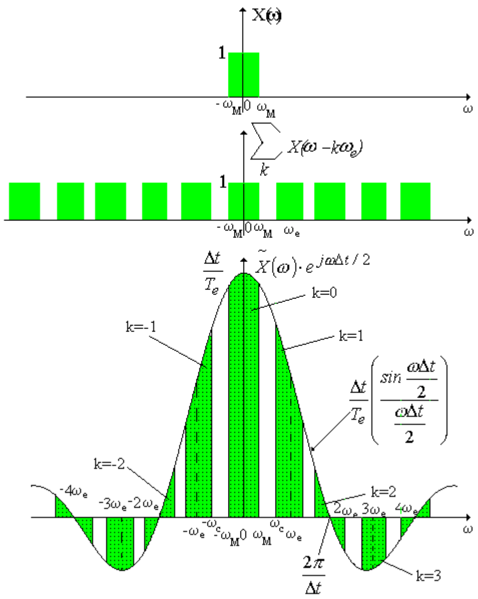
$$h(t)=p_{\frac{\Delta t}{2}}\left(t-\frac{\Delta t}{2}\right)\leftrightarrow e^{-j\frac{\omega\Delta t}{2}}\frac{2\sin\frac{\omega\Delta t}{2}}{\omega}=e^{-j\frac{\omega\Delta t}{2}}\Delta t\frac{\sin\frac{\omega\Delta t}{2}}{\frac{\omega\Delta t}{2}}$$

Spectrul semnalului esantionat cu memorare

$$\widetilde{X}(\omega)=\frac{1}{T_e}\sum_{k=-\infty}^{\infty}e^{-j\frac{\omega\Delta t}{2}}\Delta t\frac{\sin\frac{\omega\Delta t}{2}}{\frac{\omega\Delta t}{2}}X(\omega-k\omega_e)$$

$$\widetilde{X}(\omega)e^{j\frac{\omega\Delta t}{2}}=\sum_{k=-\infty}^{\infty}\frac{\Delta t}{T_e}\frac{\sin\frac{\omega\Delta t}{2}}{\frac{\omega\Delta t}{2}}X(\omega-k\omega_e)=$$

$$=\sum_{k=-\infty}^{\infty}\frac{\Delta t}{T_e}\frac{\sin\pi\frac{\Delta t}{T_e}}{\pi\frac{\Delta t}{T_e}}X(\omega-k\omega_e)$$

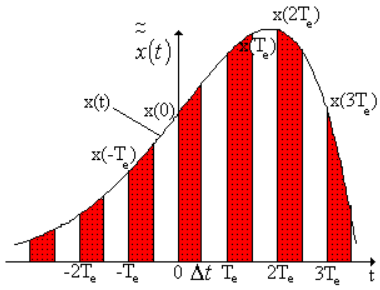


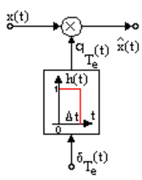
Pentru a limita erorile ce afecteaza lobul spectral central este necesar sa avem:

$$\frac{2\pi}{\Delta t} \gg \omega_M$$

ceea ce implica scurtarea duratei Δt a impulsurilor. Reconstructia prin extrapolare de ordinul zero este un caz particular PAM cu $\Delta t = T_e$.

Esantionarea naturala





$$\tilde{\tilde{x}}(t) = x(t)q_{T_e}(t) = x(t)\left[h(t)*\delta_{T_e}(t)\right] = x(t)\sum_{k=-\infty}^{\infty} x(t)h(t-kT_e) = \sum_{k=-\infty} x(t)h(t-kT_e)$$

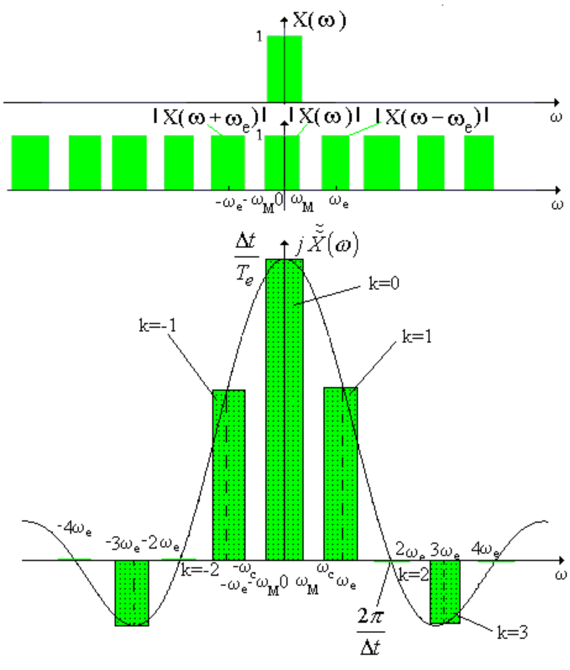
unde $h(t) = p_{\frac{\Delta t}{2}}\left(t-\frac{\Delta t}{2}\right) \leftrightarrow H(\omega) = e^{-\frac{j\omega\Delta t}{2}}\frac{2\sin\frac{\omega\Delta t}{2}}{\omega}$

Spectrul semnalului esantionat natural

$$\tilde{\tilde{X}}(\omega)=\mathcal{F}\left\{x(t)\left[h(t)*\delta_{T_e}(t)\right]\right\}=\frac{1}{2\pi}X(\omega)*\left[e^{-j\frac{\omega\Delta t}{2}}\Delta t\frac{sin\frac{\omega\Delta t}{2}}{\frac{\omega\Delta t}{2}}\frac{2\pi}{T_e}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_e)\right]$$

$$=X(\omega)*\sum_{k=-\infty}^{\infty}e^{-j\frac{k\omega_e\Delta t}{2}}\frac{\Delta t}{T_e}\frac{sin\frac{k\omega_e\Delta t}{2}}{\frac{k\omega_e\Delta t}{2}}\delta(\omega-k\omega_e)=$$

$$=\sum_{k=-\infty}^{\infty}e^{-j\frac{k\omega_e\Delta t}{2}}\frac{\Delta t}{T_e}\frac{sin\frac{k\omega_e\Delta t}{2}}{\frac{k\omega_e\Delta t}{2}}X(\omega-k\omega_e)$$



Lobii spectrali obtinuti in urma esantionarii naturale nu sunt deformati ca in cazul esantionarii cu memorare. Lobul central este asemanator cu cel obtinut la esantionarea ideala. Filtrand trece jos cu $\omega_M \le \omega_c \le \omega_e - \omega_M$ si o amplificare $T_e/\Delta t$, se poate recupera semnalul initial, $x(t)$.

$$H(\omega)=\begin{cases}\frac{T_e}{\Delta t}, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

Relatia dintre spectrul unui semnal discret si spectrul semnalului analogic din care provine

Se stie ca $\widehat{X}(\omega)=\dfrac{1}{T_e}\sum\limits_{k=-\infty}^{\infty} X_a(\omega-k\omega_e)$ *este spectrul semnalului esantionat ideal.*

$\widehat{X}(\omega)$ *se poate calcula si prin aplicarea directa a transformatei Fourier semnalului* $\hat{x}(t)$.

$$\widehat{X}(\omega)=\mathcal{F}\left\{x_a(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_e)\right\}=\mathcal{F}\left\{\sum_{k=-\infty}^{\infty}x_a(kT_e)\delta(t-kT_e)\right\}=$$

$$=\sum_{k=-\infty}^{\infty}x_a(kT_e)\mathcal{F}\{\delta(t-kT_e)\}=\sum_{k=-\infty}^{\infty}x_a(kT_e)e^{-j\omega kT_e}$$

S-au obtinut 2 expresii echivalente pentru spectrul

$$\widehat{X}(\omega)=\frac{1}{T_e}\sum_{k=-\infty}^{\infty}X_a(\omega-k\omega_e)=\sum_{k=-\infty}^{\infty}x_a(kT_e)e^{-j\omega kT_e}$$

Spectrul semnalului discret obtinut in urma esantionarii :

$$X_d(\Omega)=\sum_{k=-\infty}^{\infty}x_d[k]e^{-j\Omega k}=\sum_{k=-\infty}^{\infty}x_a(kT_e)e^{-j\Omega k}$$

Se observa ca:

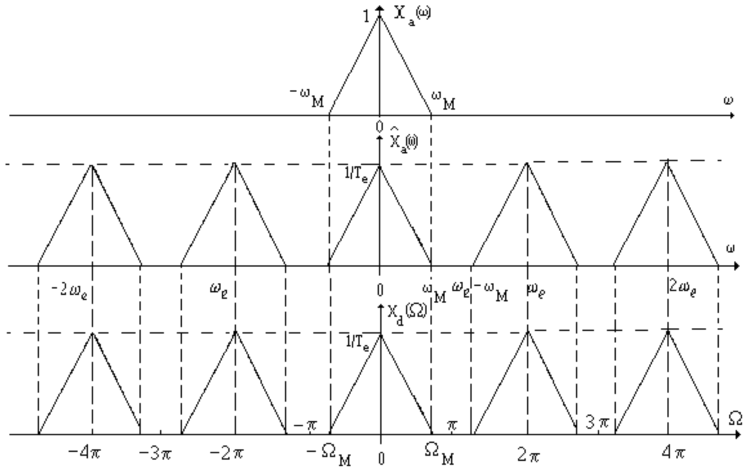
$$\left.\sum_{k=-\infty}^{\infty}x_a(kT_e)e^{-j\omega kT_e}\right|_{\omega=\frac{\Omega}{T_e}}=\sum_{k=-\infty}^{\infty}x_a(kT_e)e^{-j\Omega k}$$

si deci :

$$X_d(\Omega)=\left.\frac{1}{T_e}\sum_{k=-\infty}^{\infty}X_a(\omega-k\omega_e)\right|_{\omega=\frac{\Omega}{T_e}}=\frac{1}{T_e}\sum_{k=-\infty}^{\infty}X_a\left(\frac{\Omega}{T_e}-k\frac{2\pi}{T_e}\right)$$

adica :

$$X_d(\Omega)=\frac{1}{T_e}\sum_{k=-\infty}^{\infty}X_a\left(\frac{\Omega}{T_e}-k\frac{2\pi}{T_e}\right)$$



Intre cele doua axe de frecventa corespunzatoare spectrului semnalului analogic esantionat respectiv spectrului semnalului discret exista relatia: $\Omega = \omega T_e$. Se explica acum si natura periodica a spectrului semnalului discret $X_d(\Omega)$. Intre Ω_M si ω_M exista relatia: $\Omega_M = \omega_M T_e$; $T_e \leq \frac{\pi}{\omega_M}$.

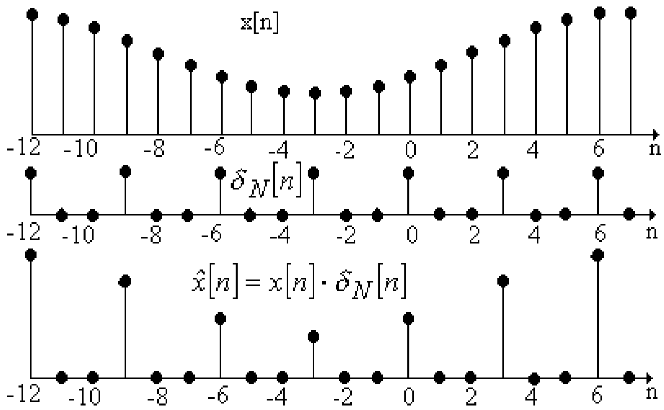
Esantionarea semnalelor discrete

In prelucrarea numerica a semnalelor apar situatii in care, ulterior achizitionarii esantioanelor, se constata ca frecventa de esantionare a fost prea mare. In astfel de situatii, cand nu se mai poate esantiona semnalul analogic, este posibila esantionarea semnalului numeric, retinandu-se tot a N-a valoare. Fie:

$$\delta_N[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

Semnalul discret esantionat, $\hat{x}[n]$, se obtine prin produsul:

$$\hat{x}[n] = x[n]\delta_N[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-kN] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN].$$



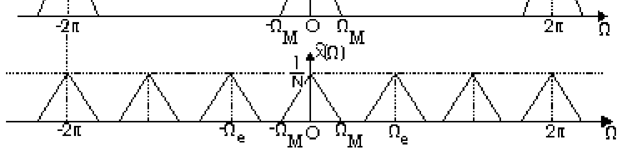
N=3.

$$\delta_N[n] \leftrightarrow \Omega_e \delta_{\Omega_e}(\Omega) = \Omega_e \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_e),\ \Omega_e = \frac{2\pi}{N}.$$

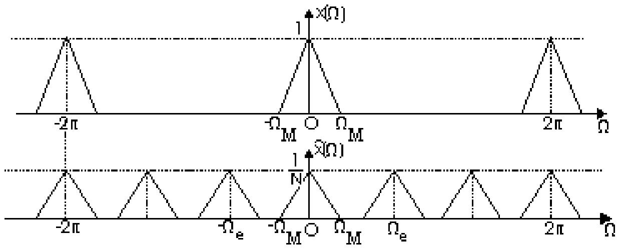
$$\widehat{X}(\Omega) = \mathcal{F}\{x[n]\delta_N[n]\} = \frac{1}{2\pi} X(\Omega) \otimes \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_e)\ ;\Omega_e=\frac{2\pi}{N}.$$

Pentru calculul convolutiei circulare se recurge la restrictionarea la perioada principala a celor doi termeni, la calculul convolutiei necirculare si la prelungirea prin periodicitate a rezultatului obtinut.

$$\widehat{X}_r(\Omega) = \frac{1}{N}X_r(\Omega) * \sum_{k=0}^{N-1}\delta(\Omega-k\Omega_e) = \frac{1}{N}\sum_{k=0}^{N-1}X_r(\Omega-k\Omega_e),\ \Omega_e=\frac{2\pi}{N}$$



N=3.

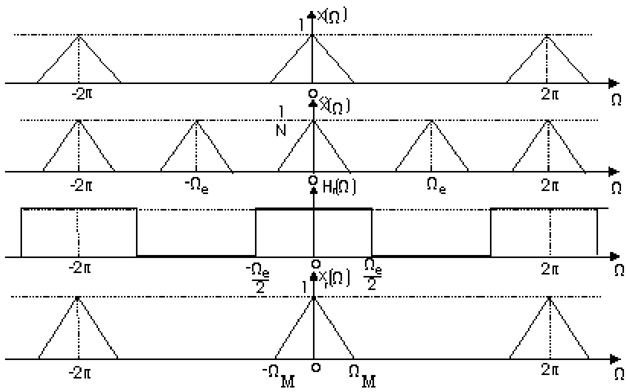


Cum $\Omega_M = T_e\omega_M$, unde ω_M este frecventa maxima din spectrul semnalului analogic din care provine $x[n]$, iar T_e pasul cu care acest semnal analogic a fost esantionat, rezulta:

$$NT_e \leq \frac{\pi}{\omega_M}\ ;\ T_e{}' \leq \frac{\pi}{\omega_M}\ ;\ T_e{}' = NT_e$$

S-ar fi respectat teorema WKS chiar daca semnalul $x(t)$ ar fi fost esantionat cu pasul $T_e{}'$. Daca $\Omega_\mathrm{e} - \Omega_M < \Omega_M$ apare suprapunerea lobilor spectrali vecini, adica erori de tip "alias".

Reconstruirea semnalului discret din esantioanele sale



$$H_r(\Omega) = \begin{cases} N, & |\Omega - 2k\pi| \leq \Omega_c \\ 0, & \text{in rest} \end{cases} \quad \Omega_M \leq \Omega_c \leq \Omega_e - \Omega_M\ .$$

Raspunsul la impuls al filtrului de reconstructie este:

$$h_r[n]=\frac{\sin n\Omega_c}{n\Omega_c}\ ;\ \Omega_c=\frac{\Omega_e}{2}=\frac{\pi}{N}\ .$$

$$x_r[n]=\hat{x}[n]*h_r[n]=x[n]\Leftrightarrow$$

$$x[n]=\sum_{k=-\infty}^{\infty}\hat{x}[k]h_r[n-k]$$

Dar $\hat{x}[k]=0$ pentru $k\neq Nm$ si $\hat{x}[Nm]=x[Nm]$ si deci

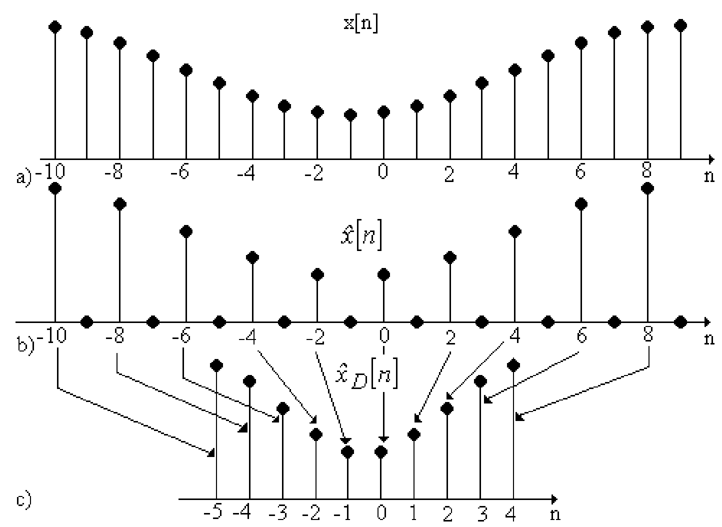
$$x[n]=\sum_{m=-\infty}^{\infty}\hat{x}[Nm]h_r[n-Nm]=\sum_{m=-\infty}^{\infty}x[Nm]\frac{\sin\left(\frac{\pi}{N}n-\pi m\right)}{\frac{\pi}{N}n-\pi m}$$

Esantionarea si decimarea unui semnal discret

Intre doua valori nenule si consecutive ale unui semnal esantionat sunt intercalate $N-1$ valori nule. Prin omiterea acestora se obtine un nou semnal, numit decimatul semnalului esantionat, care se va nota cu $\hat{x}_D[n]$. Din semnalul decimat se poate reconstrui semnalul nedecimat prin simpla inserare a cate $N-1$ zerouri intre doua valori consecutive.

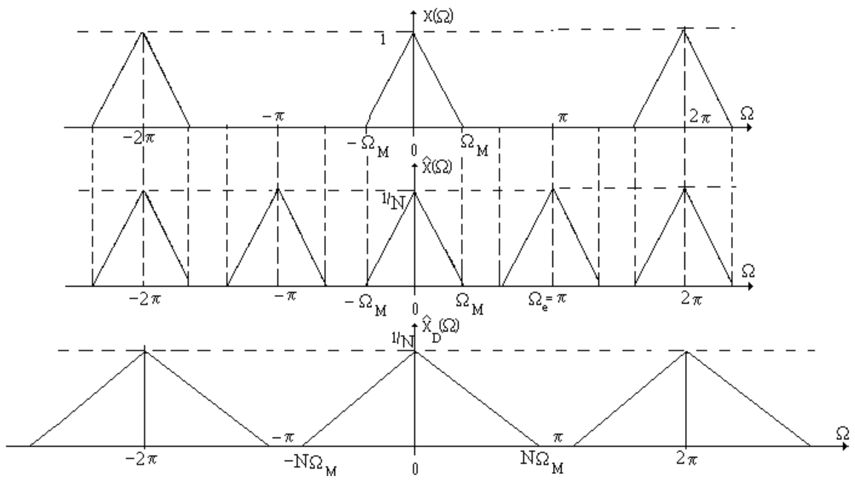
$$\widehat{X}_D(\Omega)=\mathcal{F}\left\{\hat{x}_D[n]\right\}=\sum_{n=-\infty}^{\infty}\hat{x}_D[n]e^{-j\Omega n}$$

$$=\sum_{n=-\infty}^{\infty}\hat{x}[nN]e^{-j\Omega n}=\sum_{m=-\infty}^{\infty}\hat{x}[m]e^{-j\frac{\Omega}{N}m}=\widehat{X}\left(\frac{\Omega}{N}\right).$$



Tinand seama de faptul ca spectrul semnalului esantionat reprezinta prelungirea prin periodicitate a spectrului semnalului de esantionat, relatia $\widehat{X}_D(\Omega) = \widehat{X}\left(\frac{\Omega}{N}\right)$ se mai scrie: $\widehat{X}_D(\Omega) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{\Omega - 2k\pi}{N}\right)$. In consecinta $\widehat{X}_D(\Omega - 2\pi) =$

$= \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{\Omega - (k+1)2\pi}{N}\right) \overset{k+1=l}{=} \frac{1}{N}\sum_{l=1}^{N} X\left(\frac{\Omega - l2\pi}{N}\right)$. Dar $X\left(\frac{\Omega}{N}\right) = X\left(\frac{\Omega}{N} - 2\pi\right)$ ca urmare a periodicitatii cu 2π a transformatei Fourier in timp discret. In consecinta suma din membrul drept poate fi scrisa sub forma $\frac{1}{N}\sum_{l=0}^{N-1} X\left(\frac{\Omega - l2\pi}{N}\right)$, care reprezinta tocmai $\widehat{X}_D(\Omega)$. Asadar, s-a demonstrat ca: $\widehat{X}_D(\Omega - 2\pi) = \widehat{X}_D(\Omega)$, spectrul semnalului esantionat si decimat este o functie periodica de perioada 2π. Pentru $k = 0$ se obtine lobul central, $\left(\frac{1}{N}\right) X\left(\frac{\Omega}{N}\right)$ care se anuleaza pentru $\frac{\Omega}{N} = \Omega_M$. Intinderea lobilor spectrali ai lui $\widehat{X}_D(\omega)$ este de N ori mai mare decat intinderea lobilor spectrali ai semnalului $x[n]$.



N=2.

Esantionarea spectrului unui semnal discret de durata finita

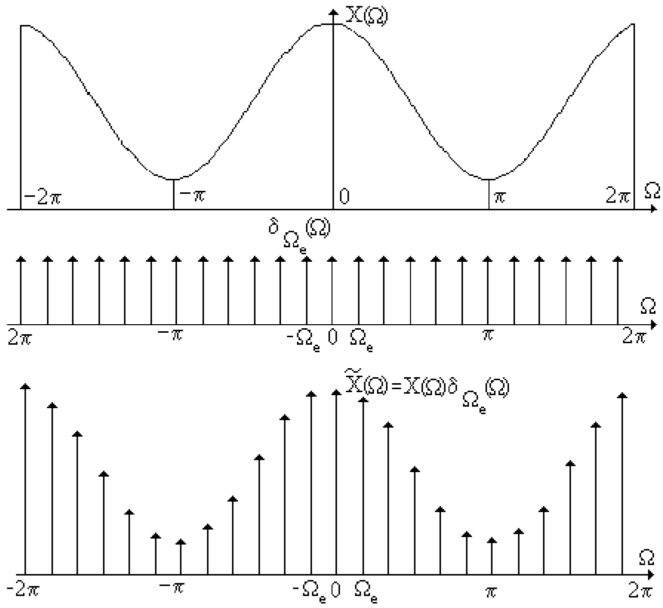
Fie $x[n]$ *un semnal aperiodic in timp discret cu spectrul* $X(\Omega)$. *Se esantioneaza ideal spectrul* $X(\Omega)$, *prin inmultire cu* $\delta_{\Omega_e}(\Omega)$. $\tilde{X}(\Omega)=X(\Omega)\delta_{\Omega_e}(\Omega)=$

$$=X(\Omega)\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_e)=\sum_{k=-\infty}^{\infty}X(k\Omega_e)\delta(\Omega-k\Omega_e).$$ *Dar* $\delta_N[n]\leftrightarrow\Omega_e\delta_{\Omega_e}(\Omega)=$

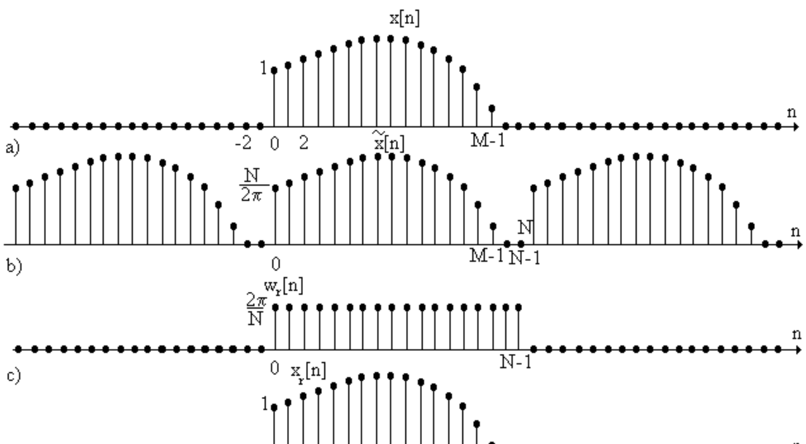
$$=\Omega_e\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_e)\ ;\ \Omega_e=\frac{2\pi}{N},$$ *de unde* $\frac{1}{\Omega_e}\delta_N[n]\leftrightarrow\delta_{\Omega_e}(\Omega)$. *De aceea*

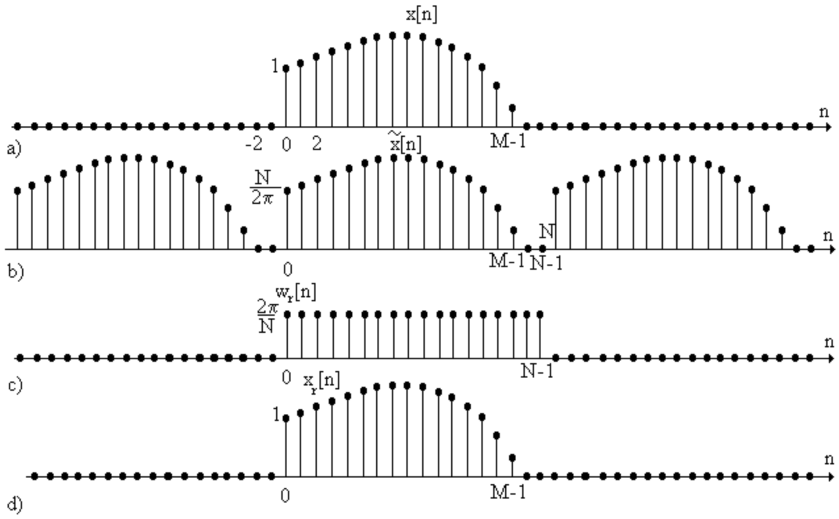
$$\tilde{x}[n]=\mathcal{F}^{-1}\left\{X(\Omega)\delta_{\Omega_e}(\Omega)\right\}=\mathcal{F}^{-1}\left\{X(\Omega)\right\}*\mathcal{F}^{-1}\left\{\delta_{\Omega_e}(\Omega)\right\}=x[n]*\frac{1}{\Omega_e}\delta_N[n]$$

sau $\tilde{x}[n]=\frac{N}{2\pi}\sum_{k=-\infty}^{\infty}x[n-kN]$, *semnalul* $\tilde{x}[n]$ *reprezinta prelungirea prin periodicitate a semnalului* $x[n]$. *Pentru ca semnalul* $x[n]$ *sa poata fi recuperat din semnalul* $\tilde{x}[n]$ *este ne esar ca suportul sau sa fie marginit.*



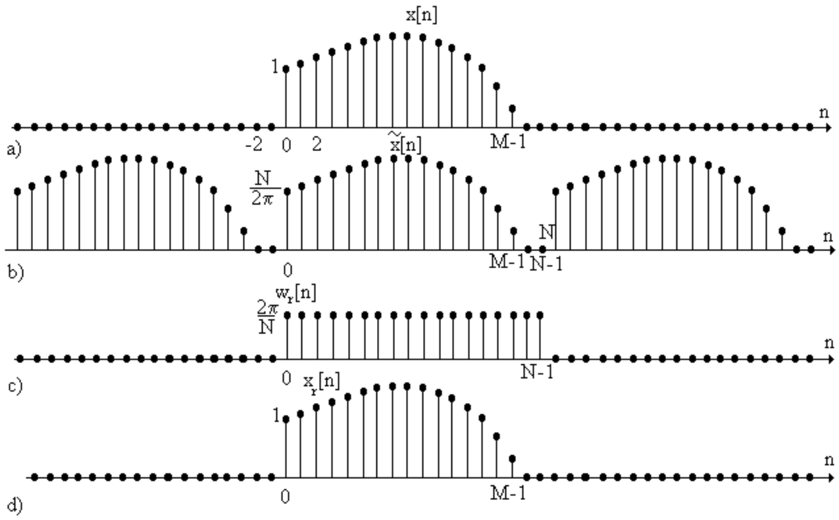
Fie $x[n]$ cu suportul $0 \le n \le M-1$. In urma esantionarii spectrului acestui semnal se obtine semnalul $\tilde{x}[n]$periodic de perioada $N = \dfrac{2\pi}{\Omega_e}$. Daca $N \ge M$ nu se produce suprapunerea grupurilor temporale corespunzatoare diverselor valori k.





Prin multiplicarea semnalului $\tilde{x}[n]$ cu fereastra temporala rectangulara $w_r[n] = \begin{cases} \frac{2\pi}{N}, & 0 \le n \le N-1 \\ 0, & \text{in rest} \end{cases}$

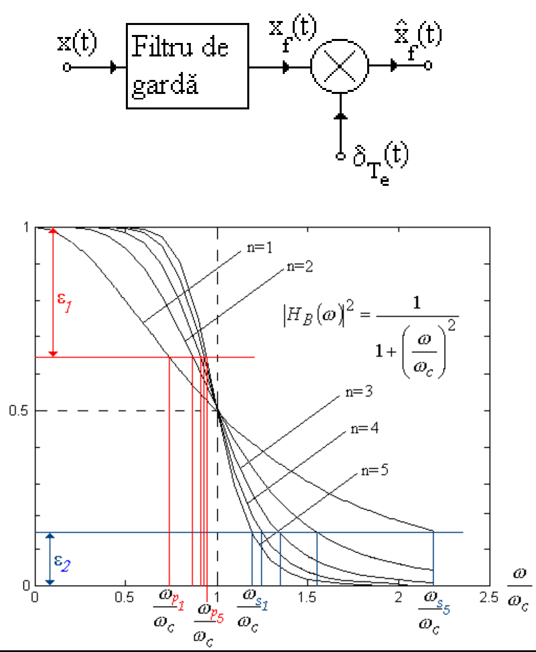
se obtine semnalul reconstruit $x_r[n]$, identic cu semnalul $x[n]$: $x_r[n] = x[n] = \tilde{x}[n] w_r[n]$.



Daca spectrul $X(\Omega)$ se esantioneaza prea rar, $M < N$, apare suprapunerea grupurilor temporale, adica erori de tip "alias". Semnalul $x[n]$ nu mai poate fi reconstruit din spectrul esantionat.

Masuri practice la esantionarea semnalelor analogice

De obicei nu se cunoaste largimea de banda a semnalului ce urmeaza a fi santionat. Acesta poate avea componente spectrale de frecventa mare, neinteresante in aplicatia considerata. Acestea pot fi de exemplu cauzate de zgomotul ce insoteste semnalul util. Exista deci riscul aparitiei erorilor de tip "alias". Pentru evitarea lor se prevede in structura lantului de prelucrare a semnalului, inaintea circuitului de esantionare, un filtru trece jos numit filtru "anti-alias" sau filtru de garda.

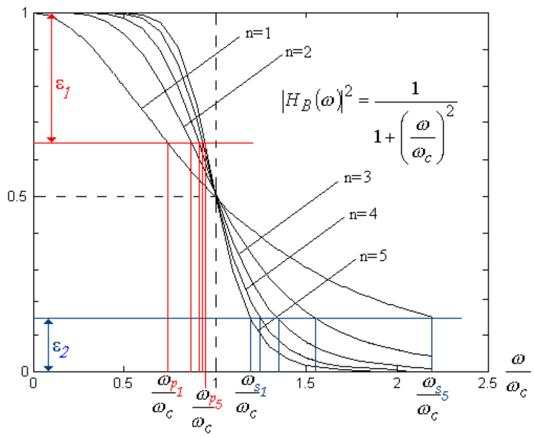


Esantionarea trebuie facuta cu o frecventa de cel putin 2 ori mai mare decat frecventa de oprire ω_s $\omega_e \geq 2\omega_s$.

De asemenea trebuie sa avem $\omega_M \leq \omega_p$. Deci:

$$\omega_M \leq \omega_p < \omega_s \leq \frac{\omega_e}{2}.$$

Cu cat banda de tranzitie $\omega_s - \omega_p$ este mai mare, cu atat frecventa de esantionare trebuie sa fie mai mare decat frecventa Nyquist $2\omega_M$.



Banda de tranzitie mai mare $\Rightarrow$ ordin de filtru mai redus, mai putine elemente constructive, mai ieftin. Cu scaderea lui ε_2 scad erorile de tip "alias" dar cresc ω_s si deci si ω_e.

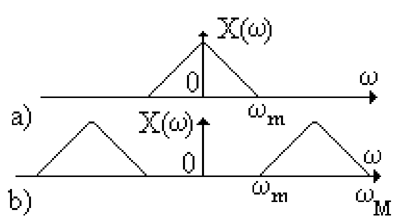
Sisteme de telefonie numerica - $f_M = 3{,}4$ KHz, $f_e = 8$ KHz.

Sisteme de televiziune - $f_M \cong 5$ MHz, $f_e = 18$ MHz.

Esantionarea semnalelor trece banda

Semnale de tip "trece jos" - spectrul concentrat in benzi care includ frecventa nula.

Semnale de tip "trece banda" - au suportul spectrului de forma $[-\omega_M, -\omega_m] \cup [\omega_m, \omega_M]$.



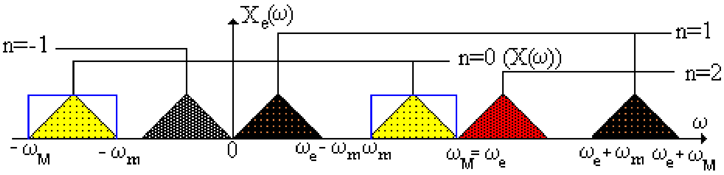
Reconstructia perfecta a unui semnal trece banda esantionat ideal se poate realiza pe baza teoremei WKS, $\omega_e \geq 2\omega_M$. Uneori semnalele trece banda pot fi reconstruite din esantioanele lor chiar daca s-a folosit o frecventa de esantionare mai mica decat frecventa Nyquist.

Cazul semnalelor trece banda de banda ingusta

$$\frac{\omega_M - \omega_m}{\omega_m} < 1.$$

Suportul spectrului unui semnal trece banda de banda ingusta esantionat ideal este de forma:

$$\text{supp}\{X_e(\omega)\} = \bigcup_{n \in Z} \{[-\omega_M + n\omega_e, -\omega_m + n\omega_e] \cup [\omega_m + n\omega_e, \omega_M + n\omega_e]\}.$$



Semnalul trece banda de banda ingusta poate fi reconstruit perfect din esantioanele sale chiar daca a fost folosita o frecventa de esantionare mai mica decat frecventa Nyquist.

Conditia de reconstructie perfecta este:

$$[-\omega_M + k\omega_e, -\omega_m + k\omega_e] \cap [\omega_m + l\omega_e, \omega_M + l\omega_e] = \varnothing,\ \forall\ k, l \in Z.$$

Pentru $l = 0$, conditia devine $[-\omega_M + k\omega_e, -\omega_m + k\omega_e] \cap [\omega_m, \omega_M] = \varnothing\ \forall k \in Z.$

adica:

$$\begin{cases} -\omega_M + k\omega_e \leq \omega_m \\ -\omega_M + (k+1)\omega_e \geq \omega_M \end{cases} \text{ sau } \frac{2\omega_M}{k+1} \leq \omega_e \leq \frac{2\omega_m}{k}.$$

Daca exista valori intregi k, pentru care aceasta conditie este satisfacuta, atunci exista valori ale frecventei de esantionare inferioare frecventei Nyquist pentru care semnalele trece banda de banda ingusta pot fi reconstruite in urma esantionarii ideale.

Solutia din multimea numerelor intregi a dublei inecuatii obtinute este: $0 < k \le \frac{\omega_m}{\omega_M - \omega_m}$. Notand cu n_0 partea intreaga a fractiei $\omega_m / \left(\omega_M - \omega_m\right)$, rezulta ca frecventa de esantionare va apartine unor intervale de forma $\left[\frac{2\omega_M}{k+1}, \frac{2\omega_m}{k}\right]$ cu $k \in \{1, ..., n_0\}$.

Exemplu

$\omega_m = 8\pi$ si $\omega_M = 10\pi$. Valoarea factorului n_0 este $\frac{\omega_m}{\omega_m - \omega_M} = 4$.

Valorile admisibile pentru k sunt 1, 2, 3 si 4. Acestor valori le corespund urmatoarele domenii pentru frecventa de esantionare:

$\{4\pi\} \cup [5\pi\ ,\ 5{,}33\pi] \cup [6{,}66\pi\ ,\ 8\pi] \cup [10\pi\ ,\ 16\pi] \cup [20\pi\ ,\ \infty]$.